



# Homological quantum error correcting codes and real projective space

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## ► To cite this version:

Vivien Londe. Homological quantum error correcting codes and real projective space. Journées Codage et Cryptographie 2017, Apr 2017, La Bresse, France. hal-01671444

**HAL Id: hal-01671444**

**<https://inria.hal.science/hal-01671444>**

Submitted on 22 Dec 2017

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# Homological quantum error correcting codes and real projective space

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April 26, 2017

# Outline

Quantum error correcting codes

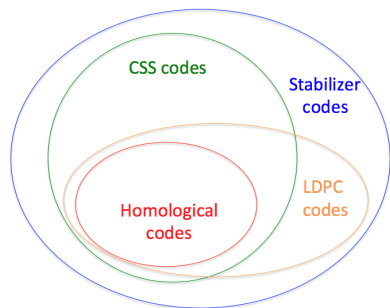
Quantum codes from real projective spaces

# Quantum error correcting codes

- ▶ qubit:  $\alpha |0\rangle + \beta |1\rangle$  where  $\alpha, \beta \in \mathbb{C}$  and  $|\alpha|^2 + |\beta|^2 = 1$ .
- ▶ two types of errors:  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .
- ▶  $[[n, k, d]]$  error correcting code: **k logical qubits**  
but **n physical qubits** ( $n > k$ )
- ▶ **minimal distance d** of the code proportional to the maximal number of **errors** which can be **corrected**.

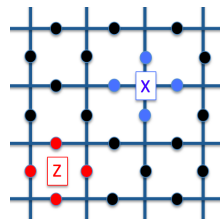
# Families of codes

- ▶ stabilizer codes: the codespace is the 1-eigenspace of a set of commuting operators.
- ▶ CSS codes: can be specified by two orthogonal linear classical codes  $\mathcal{C}_X$  and  $\mathcal{C}_Z$ .
- ▶ LDPC codes: lines and columns of parity check matrices have bounded weights.
- ▶ homological codes: The two orthogonal classical codes are defined from a cellulation of a manifold.



## Example of homological code: Toric code [Kitaev 2002]

- ▶ cellulation of the torus by squares.
- ▶ To each edge corresponds a physical bit.
- ▶ To each face corresponds a line of  $H_Z$ .
- ▶ To each vertex corresponds a line of  $H_X$ .

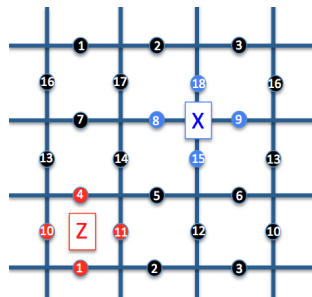


A torus is obtained by identifying left and right sides and identifying up and down sides of the square.

$\mathcal{C}_X$  and  $\mathcal{C}_Z$  are orthogonal because each (face, vertex) pair shares an even number of edges.

## Example of homological code: Toric code [Kitaev 2002]

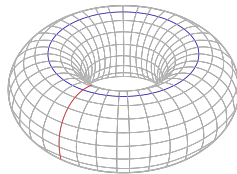
- ▶ The line of  $H_Z$  corresponding to the red face checks the parity of the bits 1, 10, 4 and 11.
- ▶ The line of  $H_X$  corresponding to the blue vertex checks the parity of the bits 8, 18, 9 and 15.



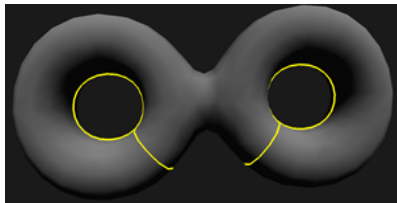
$\mathcal{C}_X$  and  $\mathcal{C}_Z$  are orthogonal because each (face, vertex) pair shares an even number of edges.

## Geometric interpretation of $k$

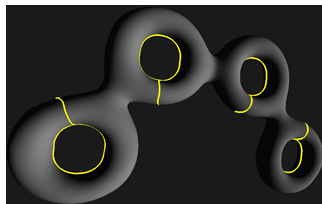
- ▶ The code dimension is the rank of the first homology group  $H_1$  of the manifold.
- ▶ Informally, it is the number of different loops of the manifold.



$$k = 2 = \text{rank}(H_1)$$



$$k = 4 = \text{rank}(H_1)$$

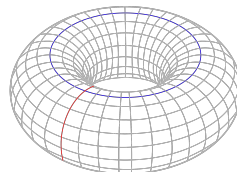


$$k = 8 = \text{rank}(H_1)$$

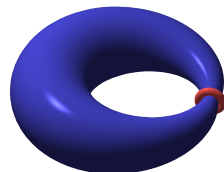


# Geometric interpretation of $n$ and $d$

- ▶ the code length  $n$  is proportional to the area of the manifold.
- ▶ the minimal distance  $d$  is proportional to the systole of the manifold.
- ▶ the systole is the length of the shortest non contractible loop of the manifold.



$$d = \text{systole} \approx 20$$
$$n = 2 \times \text{area} \approx 800$$



$$d = \text{systole} = \text{length of red circle}$$

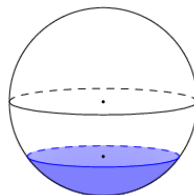
## positive curvature and the sphere $S^2$

goal: family of codes satisfying  $n = o(d^2)$

$$\text{area}(\text{disk}_{\text{euclidean}}(r)) = \pi r^2$$

$$\begin{aligned}\text{area}(\text{disk}_{\text{spherical}}(r)) &= 2\pi(1 - \cos(r)) \\ &= \pi r^2 - \frac{\pi}{12}r^4 + O(r^6)\end{aligned}$$

$$\text{area}(\text{disk}_{\text{spherical}}) < \text{area}(\text{disk}_{\text{euclidean}})$$

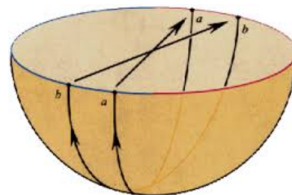


But every loop on the sphere is contractible to a point.  
Hence the dimension  $k$  of a code defined on the sphere is zero.

# The real projective plane

Identify every pair of antipodal points.  
Some loops cannot be contracted to a point.

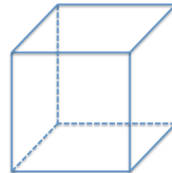
- ▶  $\text{systole}_{\text{projective plane}} = \pi$
- ▶  $\text{area}_{\text{projective plane}} = 2\pi$
- ▶  $(\text{systole}_{\text{projective plane}})^2 > \text{area}_{\text{projective plane}}$



But the area of a real projective plane of constant curvature  $+1$  is bounded above by  $2\pi$ .  
A solution is to increase dimension.

# A discrete model of the real projective plane

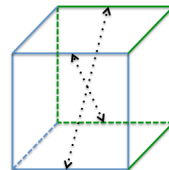
Identify pair of antipodal points of the cube.



# A discrete model of the real projective plane

Identify pair of antipodal points of the cube.

►  $n = 6$



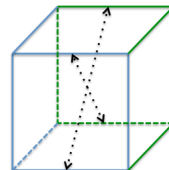
opposite edges are identified

# A discrete model of the real projective plane

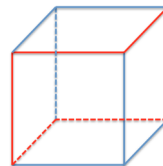
$$H_X = \text{mat}_{\text{adjacency}}(\text{vertices}, \text{edges})$$

Identify pair of antipodal points of the cube.

- ▶  $n = 6$
- ▶  $d_X = 3$



opposite edges are identified



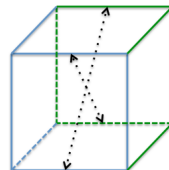
least weight codeword for the  
code  $\mathcal{C}_X$

# A discrete model of the real projective plane

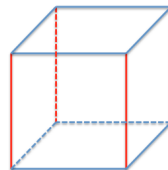
$$H_Z = \text{mat}_{\text{adjacency}}(\text{faces}, \text{edges})$$

Identify pair of antipodal points of the cube.

- ▶  $n = 6$
- ▶  $d_X = 3$
- ▶  $d_Z = 2$
- ▶  $d = \min(d_X, d_Z) = 2$



opposite edges are identified



least weight codeword for the code  $\mathcal{C}_Z$

# A discrete model of the real projective $2m$ -space

Identify pair of antipodal points of the hypercube of dimension  $2m+1$ .

Bits are identified with  $m$ -faces of the hypercube.

With the 5-hypercube:

- ▶  $n = 40$
- ▶  $d_X = 10$
- ▶  $d_Z = 4$
- ▶  $d = \min(d_X, d_Z) = 4$



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Conjecture for the  $(2m+1)$ -hypercube:

- ▶  $n = \binom{2m+1}{m} 2^m$
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- ▶  $d_Z = 2^m$

Remark:  $n = d_X d_Z$

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# Comparison with toric codes

- ▶ High dimensional projective code:
  - ▶  $d \leq \sqrt{n}$
  - ▶ By identifying bits with  $\ell$ -faces of the hypercube,  $\ell > m$ , one could reach  $d = \sqrt{n}$ .
  - ▶ logarithmically LDPC.
- ▶ High dimensional toric code:
  - ▶  $d = n^{\frac{1}{2 + \ln(4)/\ln(\ell)}}$ ,  $\ell$  is the length of the hypercube.
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- ▶ strictly LDPC.

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# Conclusions

- ▶ Positive curvature leads to greater minimal distances.
- ▶ High dimensional manifolds lead to smaller minimal distances.
- ▶ Homological quantum error correcting codes can be understood geometrically.

The only known family of codes satisfying  $d = \omega(\sqrt{n})$  is a homological code built on a four dimensional manifold with non zero curvature [Freedman 2002].

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**Thank you for your attention! Questions?**